Math 32B: Vector Potentials, $H^2(S^2 \vee S^2) \to H^2(S^2)$

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1. Consider the following vector field with domain $\mathbb{R}^3 - \{(0,0,0)\}$:

$$\mathbf{F}(x,y,z) = \frac{(x,y,z)}{x^2+y^2+z^2}.$$

   (a) What is $\nabla \cdot \mathbf{F}$?

   (b) Let S be the unit sphere $x^2+y^2+z^2 = 1$ with outward pointing normal. Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$.

   (c) Does $\mathbf{F}$ have a vector potential defined on $\mathbb{R}^3 - \{(0,0,0)\}$?

   (d) How would you justify your answer to (c) as quickly as possible?

2. Consider the following vector field with domain $\mathbb{R}^3 - \{(0,0,0)\}$:

$$\mathbf{F}(x,y,z) = \frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}.$$

   (a) What is $\nabla \cdot \mathbf{F}$?

   (b) Let S be the unit sphere $x^2+y^2+z^2 = 1$ with outward pointing normal. Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$.

   (c) Does $\mathbf{F}$ have a vector potential defined on $\mathbb{R}^3 - \{(0,0,0)\}$?

   (d) How would you justify your answer to (c) as quickly as possible?

3. Consider the following vector field with domain $\mathbb{R}^3 - \{(0,0,0)\}$:

$$\mathbf{F}(x,y,z) = \frac{(x(z^2-y^2), y(x^2-z^2), z(y^2-x^2))}{(x^2+y^2+z^2)^{\frac{3}{2}}}.$$

(a) What is $\nabla \cdot \mathbf{F}$?

(b) Let S be the unit sphere $x^2+y^2+z^2=1$ with outward pointing normal. Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$.

(c) Does $\mathbf{F}$ have a vector potential defined on $\mathbb{R}^3 - \{(0,0,0)\}$?

(d) How would you justify your answer to (c) as quickly as possible?

4. Let $\mathbf{F}_1(x,y,z) = \frac{(x,(y+1),z)}{(x^2+(y+1)^2+z^2)^{\frac{3}{2}}}$ and $\mathbf{F}_2(x,y,z) = \frac{(x,(y-1),z)}{(x^2+(y-1)^2+z^2)^{\frac{3}{2}}}$.

Their domains are $\mathbb{R}^3 - \{(0,-1,0)\}$ and $\mathbb{R}^3 - \{(0,1,0)\}$, respectively.

Let S_1 be the unit sphere $x^2+(y+1)^2+z^2=1$ with outward pointing normal. Let S_2 be the unit sphere $x^2+(y-1)^2+z^2=1$ with outward pointing normal. Let S_3 be the sphere of radius 3, $x^2+y^2+z^2=9$, with outward pointing normal.

(a) What are $\nabla \cdot \mathbf{F}_1$ and $\nabla \cdot \mathbf{F}_2$?

(b) What are $\iint_{S_1} \mathbf{F}_1 \cdot d\mathbf{S}$, $\iint_{S_2} \mathbf{F}_2 \cdot d\mathbf{S}$, $\iint_{S_1} \mathbf{F}_2 \cdot d\mathbf{S}$, $\iint_{S_2} \mathbf{F}_1 \cdot d\mathbf{S}$?

(c) What are $\iint_{S_3} \mathbf{F}_1 \cdot d\mathbf{S}$ and $\iint_{S_3} \mathbf{F}_2 \cdot d\mathbf{S}$?
Hint: use local surface-independence.

(d) Consider $\mathbf{F} = \mathbf{F}_1 - \mathbf{F}_2$ with domain $\mathbb{R}^3 - (\{(0,-1,0),(0,1,0)\}$.
What is $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$?

Does $\mathbf{F}$ have a vector potential on $\mathbb{R}^3 - (\{(0,-1,0),(0,1,0)\}$?

(e) Consider $\mathbf{G} = \mathbf{F}_1 - \mathbf{F}_2$ with domain $\mathbb{R}^3 - \{0\} \times [-1,1] \times \{0\}$.
What is $\iint_{S_3} \mathbf{G} \cdot d\mathbf{S}$?

Does $\mathbf{G}$ have a vector potential on $\mathbb{R}^3 - \{0\} \times [-1,1] \times \{0\}$?

Solutions

1. (a) $\frac{1}{x^2+y^2+z^2}$.
   (b) 4π.
   (c) No.
   (d) Part (b) shows it has a non-zero flux.
2. (a) 0.
   (b) 4π.
   (c) No.
   (d) Part (b) shows it has a non-zero flux.
3. (a) 0.
   (b) 0.
   (c) Yes.
   (d) It has divergence 0 and all potentially non-zero fluxes are 0.
   Also, $\mathbf{A} = \frac{(yz,zx,xy)}{\sqrt{x^2+y^2+z^2}}$ is a vector potential for $\mathbf{F}$.
   I don't expect you to be able to find the vector potential.
4. (a) 0, 0.
   (b) 4π, 4π, 0, 0.
   (c) 4π, 4π.
   (d) 4π. No: it has a non-zero flux.
   (e) 0. Yes: it has divergence 0 and all potentially non-zero fluxes are 0; also, we have a vector potential for $\mathbf{F}$ defined by $\mathbf{A}(x,y,z) = \frac{(y+1)(-z,0,x)}{(x^2+z^2)\sqrt{x^2+(y+1)^2+z^2}} - \frac{(y-1)(-z,0,x)}{(x^2+z^2)\sqrt{x^2+(y-1)^2+z^2}}$ when $x^2+z^2 \neq 0$, and $\mathbf{A}(0,y,0) = 0$ when $|y| > 1$.
   I don't expect you to be able to find the vector potential.